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LOCAL-GLOBAL PRINCIPLES FOR ALGEBRAIC COVERS

BY

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ABSTRACT

This paper is devoted to some local-global type questions about fields of definition of algebraic covers. Let $f\colon X \to B$ be a cover *a priori* defined over $\overline{\mathbb{Q}}$. Assume that the cover f can be defined over each completion $\mathbb{Q}_p$ of $\mathbb{Q}$. Does it follow that the cover can be defined over $\mathbb{Q}$? This is the *local-to-global* principle. It was shown to hold for G-covers [DbDo], i.e., for Galois covers given with their automorphisms. Here we prove that, in the situation of *mere covers*, the local-to-global principle holds under some additional assumptions on the group G of the cover and the monodromy representation $G \hookrightarrow S_d$ (with $d = \deg(f)$). This local-to-global problem is closely related to the obstruction to the field of moduli being a field of definition. This problem was studied in [DbDo], which is the main tool of the present paper.

1. Presentation

1.1. THE LOCAL-TO-GLOBAL PROBLEM. Let B be an algebraic variety defined over a number field K and $f\colon X \to B$ be a cover *a priori* defined over $\overline{K}$. Assume that the cover f can be defined over each completion K_v of K. Does it follow that the cover can be defined over K? We say that the *local-to-global* principle holds when the answer is "Yes". More generally, the same problem can be considered with the base field K a field with a proper set M_K of places satisfying the product formula. However, the local-to-global principle obviously fails if the following condition does not hold:

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(*) The field K is the only finite extension of K which can be embedded in K_v for all places $v \in M_K$.

Indeed, if k is a proper extension of K that can be embedded in all K_v's, then a cover defined over k but not on K yields a counter-example to the local-to-global principle. We will assume that condition (*) holds. Classically, that is the case for number fields and for rational function fields $\overline{k}(T)$ in one indeterminate over an algebraically closed field.

Assumption (*) guarantees that if a cover is defined over each K_v, then its *field of moduli* is contained in K. Indeed, the field of moduli of a cover (§2.5) is the smallest possible field of definition, so is contained in each field of definition. However, the field of moduli need not be a field of definition. But when that is the case, the local-to-global principle obviously holds. For example, the field of moduli is a field of definition when the cover has no automorphisms (Fried [Fr]), or, when the cover is a Galois cover of $\mathbb{P}^1$ (Coombes–Harbater [CoHa]). The real problem is when the field of moduli *is not a priori* a field of definition.

The local-to-global problem was raised by E. Dew [Dew] for G-covers of $\mathbb{P}^1$ and for K a number field. A G-cover is the data consisting of a Galois cover given together with its automorphisms. In the sequel, we use the phrase "mere covers" for non-necessarily Galois covers given *without* their automorphisms. Dew conjectured in particular that the local-to-global principle holds for G-covers of $\mathbb{P}^1$ over number fields. This was proved in [Db] except for number fields that are exceptions to Grunwald's theorem (the field $\mathbb{Q}$ is not exceptional). This result was extended to G-covers of a general base space B in [DbDo].

This paper is aimed at extending these results. The main direction is to consider the local-to-global principle for *mere covers*. We also systematically consider the situation where the base space B is an arbitrary algebraic variety B and K is a more general field. These generalizations give rise to new difficulties that we explain below.

1.2. Main ingredients. The problem is closely related to the obstruction to the field of moduli being a field of definition. This is a classical problem, which was studied in quite a general way in [DbDo]: the Main Theorem of [DbDo] is a pure cohomological characterization of the obstruction. This will be the main tool of the paper.

1.2.1. Condition (Seq/Split): Essentially, the case that the base space B is the projective line $\mathbb{P}^1$ is easier because $\mathbb{P}^1$ has many K-rational points. This implies that the exact sequence of arithmetic fundamental groups

$$1 \to \Pi_{K_s}(B^*) \to \Pi_K(B^*) \to G(K) \to 1$$

is split. Here, given a field F over which B and the branch locus are defined, B^* denotes the space B with the branch locus removed and $\Pi_F(B^*)$ the F-arithmetic fundamental group of B^*; also K_s denotes the separable closure of K and $G(K) = G(K_s/K)$ the absolute Galois group of K. That splitting condition will be denoted by (Seq/Split) as in [DbDo]. It is a natural simplifying assumption but may not hold in general. That is the main difficulty of the case of an arbitrary base space B. But an important achievement of [DbDo] is precisely to have dealt with the unsplit case.

1.2.2. Main obstruction to the field of moduli being a field of definition: [DbDo] showed that the mere cover case is much more difficult than the G-cover case. For a G-cover of group G, the obstruction to the field of moduli being a field of definition corresponds to the vanishing of *a single* 2-cocycle Ω in $H^2(K, Z(G))$ with values in the center $Z(G)$ of G and with trivial action of $G(K)$ on $Z(G)$, while for a mere cover, the obstruction corresponds to the vanishing of *at least one out of several* 2-cocycles $(\Omega_\Lambda)_{\Lambda\in\Delta}$ in $H^2(K, Z(G), L)$ (for a *non-necessarily trivial* action L on $Z(G)$). Here G is the Galois group of the Galois closure of the cover f (or, equivalently, the monodromy group of f in characteristic 0).

1.2.3. First obstruction to the field of moduli being a field of definition (condition (λ/Lift)): Furthermore, in the case of mere covers, the index set Δ may be empty (in which case, of course, the field of moduli is not a field of definition). That is an additional obstruction, which does not exist for G-covers and which is called the *first obstruction* in [DbDo]. It is shown to correspond to the (weak) solvability of an embedding problem for the absolute Galois group $G(K)$ (see Proposition 3.1 (or [DbDo; §3.1]) for more details). This condition is called (λ/Lift) in [DbDo]. The various solutions to this embedding problem correspond to the elements Λ of Δ.

Concretely, elements Λ of Δ can be interpreted as follows. A quite significant invariant of the K-models of a cover is the extension $\widehat{K}/K$ of constants in the Galois closure $\widehat{f}$ of the cover f (§2.4). By definition, this invariant is trivial for G-covers: all K-models of a G-cover are regular and Galois over K, whereas mere covers may have several models with essentially distinct extensions of constants in Galois closure. These several extensions of constants correspond to the several elements $\Lambda \in \Delta$, which parametrize the several obstructions Ω_Λ.

1.3. SKETCH OF THE METHOD. This two-level obstruction led us to share the local-to-global problem in several sub-questions, also of local-global type. The first one considers the local-to-global principle with fixed extension of constants in Galois closure, i.e., with fixed $\Lambda \in \Delta$. More precisely, an extension $\widehat{K}/K$ is given and for all $v \in M_K$, the cover is assumed to be defined over K_v with $\widehat{K}K_v/K_v$ as extension of constants in Galois closure. This question is the topic of §3.2. The main conclusion (Proposition 3.3) is that the obstruction to K being a field of definition lies in the kernel of the map

$$\text{(LocGlob)} \qquad H^2(K, Z(G), L) \to \prod_{v \in M_K} H^2(K_v, Z(G), L).$$

Injectivity of this map only depends on the field K, the group $Z(G)$ and the action L (which is explicitly described in [DbDo]). For example, this map is injective if $K = \mathbb{Q}$ and L is the trivial action. Other examples are given in §3 (Proposition 3.4).

In §3.3 we let Λ vary in Δ, only assuming that $\Delta \neq \emptyset$, i.e., that condition (λ/Lift) holds. We first explain that the possibility that card(Δ) > 1 makes the local-to-global principle very unlikely in general. We however do not have any counter-example yet. In [DbDo] a quite precise description of the set Δ is given in terms of the monodromy representation $G \hookrightarrow S_d$ (with $d = \deg(f)$). Some additional assumptions then insure that all the obstructions Ω_Λ ($\Lambda \in \Delta$) are the same, in which case the same techniques as in §3.2, where Λ is fixed, can be used.

§4 is concerned with the remaining condition (λ/Lift). If a cover is defined over each K_v, then condition (λ/Lift) automatically holds over each K_v ($v \in M_K$) (Proposition 3.1). The main conclusion of §4 (Theorem 4.2) is that, under some assumptions on the representation $G \hookrightarrow S_d$, the obstruction to condition (λ/Lift) to hold globally (i.e., over K) lies in the kernel of a map similar to the map (LocGlob) above but with $Z(G)$ replaced by $Z(C/Z(G))$, where $C = \mathrm{Cen}_{S_d} G$ is the centralizer of the group G in the representation $G \hookrightarrow S_d$. Here again, some concrete situations for which the obstruction completely vanishes will be given.

1.4. MAIN RESULTS. The following result recapitulates §3–§5. Assumptions on K and B and notation are as above. In particular, condition (*) is assumed.

Condition (λ/Lift) involved was introduced in §1.2.3 and is developed in §3.1.

THEOREM: *Suppose we are given a mere cover $f\colon X \to B$ defined over each completion K_v of K ($v \in M_K$). Let G be the group of the cover and $G \hookrightarrow S_d$ be the monodromy representation. Assume in addition that the center $Z(G)$ is a direct summand of $C = \mathrm{Cen}_{S_d} G$.*

Then the field K is the field of moduli of the mere cover f and the obstruction to K being a field of definition is the following two-level obstruction:

(1st) *The first obstruction is the obstruction for condition* (λ/Lift), *which holds over each K_v ($v \in M_K$), to hold over K.*

(Main) *If condition (λ/Lift) holds over K, then the main obstruction to K being a field of definition corresponds to the vanishing of a 2-cocycle $\Omega \in H^2(K, Z(G), L)$, which lies in the kernel of the map* (LocGlob) *above.*

The *1st* part corresponds to Proposition 3.1 and the *Main* part to Theorem 3.7(a). Injectivity of the map (LocGlob) is investigated in Proposition 3.4: it is injective in particular when $K = \mathbb{Q}$ and $Z(G) \subset Z(N)$ with $N = \mathrm{Nor}_{S_d} G$. Finally, we show (Theorem 4.2) that the *1st* obstruction above vanishes (and so condition (λ/Lift) holds over K) if the three conditions (iii/1-3) below hold. From Proposition 3.1(b) of [DbDo], condition (λ/Lift) can also be guaranteed by condition (iii)$'$ below. Thus we obtain this concrete application.

THEOREM: *Assume that $K = \mathbb{Q}$, or, more generally, that K is a number field for which the special case of Grunwald's theorem cannot occur. Then the local-to-global principle holds for mere covers satisfying simultaneously these five conditions:*

(i) *$Z(G) \subset Z(N)$ where $N = \mathrm{Nor}_{S_d} G$,*

(ii) *$Z(G)$ is a direct summand of $C = \mathrm{Cen}_{S_d} G$,*

(iii/1) *$Z(C/Z(G))$ is a direct summand of $C/Z(G)$,*

(iii/2) *$Z(CG/G) \subset Z(N/G)$,*

(iii/3) *$\mathrm{Inn}(C/Z(G))$ has a complement in $\mathrm{Aut}(C/Z(G))$.*

These five conditions hold, for example, if $Z(G) = C \subset Z(N)$.

The local-to-global principle also holds for mere covers satisfying simultaneously conditions (i), (ii) *above and the following condition:*

(iii)$'$ *CG/G has a complement in N/G.*

A final section ends the paper, which is concerned with the following *global-to-local* principle (Theorem 5.1). A mere cover (or a G-cover) $f\colon X \to B$ defined over $\overline{\mathbb{Q}}$ with a number field K as field of moduli is necessarily defined over all but finitely many completions K_v of K.

1.5. OPEN QUESTIONS. There remain many interesting open questions. For example, we do not have any counter-example to the local-to-global principle for G-covers over a number field in the special case of Grunwald's theorem. Also, although we suspect that the local-to-global principle does not hold for mere covers in general, we do not have any counter-example. Finally, for most of our applications, we make the action L on the center $Z(G)$ equal to the trivial action. That is the main situation where we can prove that the local-global map (LocGlob) is injective. Investigating injectivity of the map (LocGlob) when the action is not the trivial one would be worthwhile.

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2. Preliminaries

From now on, fix a field K and a regular projective geometrically irreducible variety B defined over K. We recall below some basics relative to covers, G-covers, arithmetic fundamental spaces and the dictionary between covers and representations of fundamental groups. For more details we refer, for example, to [DbDo; §2].

2.1. NOTATION. Given a Galois extension E/k, its Galois group is denoted by $G(E/k)$. Given a field k, we denote by k_s a separable closure of k and by $G(k)$ the absolute Galois group $G(k_s/k)$ of k. As usual in Galois cohomology, we write $H^n(k,-,-)$ for $H^n(G(k),-,-)$.

2.2. MERE COVERS AND G-COVERS. A mere cover of B over K is a finite and generically unramified morphism f: $X \to B$ defined over K with X a normal and geometrically irreducible variety. A G-cover of B of group G over K is a Galois cover f: $X \to B$ over K given together with an isomorphism h: $G \to G(K(X)/K(B))$. An isomorphism between two mere covers f: $X \to B$ and f: $X' \to B$ over K is an algebraic isomorphism χ: $X \to X'$, defined over K and such that $\chi \circ f' = f$. An isomorphism of G-covers of group G over K is an isomorphism of mere covers that commutes with the given actions of G.

A mere cover $f\colon X \to B$ over a separably closed field k has two basic geometric invariants, which only depend on the isomorphism class of the cover. First the *group* G of the cover, i.e., the automorphism group of the Galois closure $\hat{f}\colon \widehat{X} \to B$ of f, or, equivalently, the Galois group $G(k(\widehat{X})/k(B))$. In characteristic 0, the group G is also the monodromy group of the cover. Second, the *branch locus* D of the cover; from the Purity of Branch Locus (e.g. [Mi]), it is a divisor of B with only simple components. By invariants of a cover over a non-separably closed field K, we always mean the invariants of the cover over K_s obtained by extension of scalars. The branch locus D of a cover over K is invariant under the action of $G(K)$.

The variety $B-D$ is denoted by B^*. If F is any field containing K, the F-arithmetic fundamendal group of B^* is denoted by $\Pi_F(B^*)$, or simply by Π_F when the context is clear. Degree d mere covers of B over F with branch locus in D correspond to transitive representations

$$\Psi\colon \Pi_F(B^*) \to S_d$$

such that the restriction to $\Pi_{F_s}(B^*)$ is transitive. G-covers of B of group G over F correspond to surjective homomorphisms

$$\Phi\colon \Pi_F(B^*) \to G$$

such that $\Phi(\Pi_{F_s}(B^*)) = G$.

2.3. Descent of the field of definition. As in [DbDo], we frequently use the word "(G-)cover" for the phrase "mere cover (resp. G-cover)". Suppose we are given a (G-)cover $f\colon X \to B$ *a priori* defined over K_s such that the branch locus D is $G(K)$-invariant. Descent of the field of definition of the (G-)cover f can be handled simultaneously for both the mere cover and G-cover situations.

In both cases let G denote the group of the cover. Then set

$$N = \begin{cases} G & \text{in the } G\text{-cover case} \\ \mathrm{Nor}_{S_d} G & \text{in the mere cover case} \end{cases}$$

and

$$C = \mathrm{Cen}_N G = \begin{cases} Z(G) & \text{in the } G\text{-cover case} \\ \mathrm{Cen}_{S_d} G & \text{in the mere cover case} \end{cases}$$

where $Z(G)$ is the center of G and $\mathrm{Nor}_{S_d} G$ and $\mathrm{Cen}_{S_d} G$ are respectively the normalizer and the centralizer of G in S_d. Finally, regard N as a subgroup of

S_d where d is the degree of f: in the mere cover case, an embedding $N \hookrightarrow S_d$ is given by definition; in the G-cover case, embed $N = G$ in S_d by the regular representation of G.

Then, in both the mere cover and G-cover situations, we have the following:

(1)(a) the (G-)cover $f\colon X \to B$ corresponds to a homomorphism (or representation)

$$\phi\colon \Pi_{K_s}(B^*) \twoheadrightarrow G \subset N,$$

(b) the (G-)cover f can be defined over the field K if and only if the homomorphism $\phi\colon \Pi_{K_s}(B^*) \to G \subset N$ can be extended to a homomorphism $\Pi_K(B^*) \to N$,

(c) two (G-)covers over K_s are isomorphic if and only if the corresponding representations ϕ and ϕ' are conjugate by an element φ in the group N, that is,

$$\phi'(x) = \varphi\phi(x)\varphi^{-1} \quad \text{for all } x \in \Pi_{K_s}(B^*).$$

2.4. Extension of constants in the Galois closure. Assume that the (G-)cover f can be defined over K, i.e., has a model f_K over K. Let $\phi_K\colon \Pi_K \to N$ be the associated extension of ϕ to $\Pi_K(B^*)$. Consider the function field extension $K(X_K)/K(B)$ associated to f_K. Denote the Galois closure of the extension $K(X_K)/K(B)$ by $\widehat{K(X_K)}/K(B)$.

Consider then the field $\widehat{K} = \widehat{K(X_K)} \cap K_s$. The extension $\widehat{K}/K$ is called the **extension of constants in the Galois closure** of the model f_K of f.

Denote by Λ the unique homomorphism $G(K) \to N/G$ that makes the following diagram commute. Existence of Λ follows from $\phi_K(\Pi_{K_s}) = \phi(\Pi_{K_s}) \subset G$ and uniqueness from the surjectivity of $\Pi_K \to G(K)$.

$$\begin{array}{ccc} \Pi_K(B^*) & \longrightarrow & G(K) \\ {\scriptstyle \phi_K}\downarrow & & \downarrow{\scriptstyle \Lambda} \\ N & \longrightarrow & N/G \end{array}$$

The homomorphism $\Lambda\colon G(K) \to N/G$ corresponds to the extension of constants $\widehat{K}/K$ in the Galois closure of the model f_K of f_F. That is, $G(F/\widehat{K}) = \mathrm{Ker}(\Lambda)$ (e.g. [DbDo; Prop. 2.3]). The homomorphism $\Lambda\colon G(K) \to N/G$ is called the **constant extension map (in Galois closure)** of the K-model f_K of f. For G-covers, $N/G = \{1\}$, the map Λ is trivial and $\widehat{K} = K$: by definition, G-covers over K are required to be Galois over K with the same Galois group as over K_s; thus they do not have any extension of constants in their Galois closure.

2.5. FIELD OF MODULI. As above, let $f\colon X \to B$ be a mere cover (resp. G-cover) *a priori* defined over K_s. For each $\tau \in G(K)$, we let $f^\tau\colon X^\tau \to B^\tau$ denote the corresponding conjugate (G-)cover. Consider the subgroup $M(f)$ (resp. $M_G(f)$) of $G(K)$ consisting of all the elements $\tau \in G(K)$ such that the covers (resp., the G-covers) f and f^τ are isomorphic over K_s. Then the **field of moduli** of the cover f (resp., the G-cover f) is defined to be the fixed field

$$K_s^{M(f)} \text{ (resp. } K_s^{M_G(f)}\text{)}$$

of $M(f)$ (resp. $M_G(f)$) in K_s. The field of moduli of a (G-)cover is easily seen to be a finite extension of K contained in each field of definition containing K. So it is the smallest field of definition containing K provided that *it is* a field of definition. The branch locus D of f is automatically invariant under $M(f)$ (resp. $M_G(f)$).

3. The local-to-global principle

In this section, we assume that the field K is a field with the product formula satisfying condition (*) of §1. Fix a G-cover or a mere cover, i.e., according to the terminology of §2.3, a (G-)cover $f\colon X \to B$ *a priori* defined over K_s and let $\phi\colon \Pi_{K_s}(B^*) \twoheadrightarrow G \subset N$ be the corresponding representation.

3.1. POSITION OF THE PROBLEM. We assume that

(Loc) The (G-)cover f can be defined over each completion K_v of K, i.e., has a model f_v over K_v ($v \in M_K$).

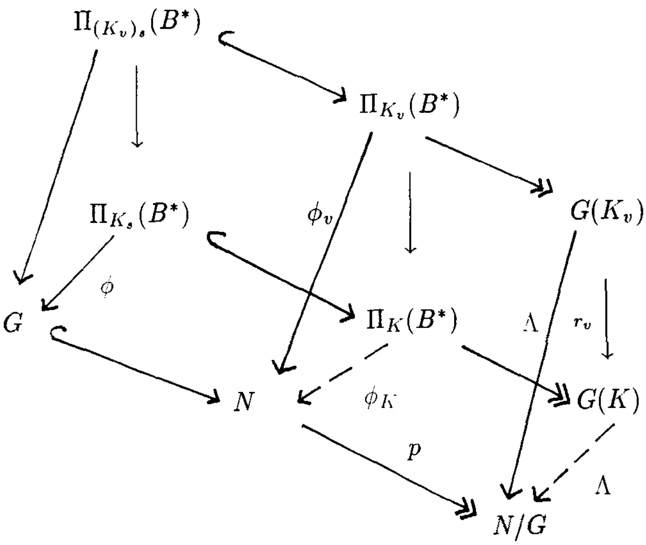
The problem is whether this local hypothesis implies the following global one:

(Glob) The (G-)cover f can be defined over K.

For each place $v \in M_K$, denote by $\phi_v\colon \Pi_{K_v}(B^*) \twoheadrightarrow G \subset N$ the representation corresponding to the model f_v over K_v. That f_v is a model of f over K_v means that restrictions of ϕ and ϕ_v to $\Pi_{(K_v)_s}(B^*)$ are conjugate by an element $\varphi_v \in N$† (in the sense of (1)(c) of §2). For each place $v \in M_K$, we have a diagram like that below. The problem consists in studying whether the map ϕ can be extended to

† This does not depend on the embeddings $K_s \subset (K_v)_s$ and $K_v \subset (K_v)_s$.

a map $\phi_K\colon \Pi_K(B^*) \to N$.



Each such extension $\phi_K\colon \Pi_K(B^*) \to N$ of ϕ should induce over $G(K)$ a map $\Lambda\colon G(K) \to N/G$, namely, the constant extension map (in Galois closure) of the K-model corresponding to ϕ_K. Existence of such maps Λ should be questioned first. Indeed, [DbDo] revealed some constraint. As we have already noted (§1.1), hypothesis (Loc) implies that K is the field of moduli of the (G-)cover f. Under the latter condition, to the (G-)cover f is attached a homomorphism $\lambda\colon G(K) \to N/CG$ with the following property [DbDo; §3.1].

Proposition 3.1:

(a) *The constant extension map Λ: $G(K) \to N/G$ of each K-model of the (G-)cover f is a lifting of λ. In particular, condition*

(λ/Lift) *There exists at least one lifting Λ: $G(K) \to N/G$ of λ: $G(K) \to N/CG$.*

is a necessary condition for the field of moduli K to be a field of definition of the (G-)cover.

(b) *Under hypothesis* (Loc), *condition* (λ/Lift) *necessarily holds over K_v, for each $v \in M_K$, that is, there exists a group homomorphism Λ_v: $G(K_v) \to N/G$ that lifts the map λ_v: $G(K_v) \to N/CG$ obtained by composing λ with the natural map r_v: $G(K_v) \to G(K)$.*

The homomorphism λ: $G(K) \to N/CG$ is precisely defined in [DbDo], where it is called *the constant extension map (in Galois closure) modulo C* given by the field of moduli condition. It is uniquely determined by the representation ϕ: $\Pi_{K_s}(B^*) \twoheadrightarrow G \subset N$. Condition ($\lambda$/Lift) is presented in [DbDo] as the *first obstruction* to the field of moduli being a field of definition. In the case of G-covers, we have $N/CG = N/G = \{1\}$. Thus condition (λ/Lift) holds trivially. The case of mere covers is different. The map λ may have no liftings Λ (i.e., condition (λ/Lift) may not hold), and may have several ones.

Proof of Proposition 3.1: (a) corresponds to Main Theorem (I) of [DbDo]. It follows from the definition of λ [DbDo;Main Theorem (I)(a)] that it behaves well with extensions of scalars: in particular, for each $v \in M_K$, the constant extension map modulo C of the cover f_v over K_v is $\lambda_v = \lambda r_v$; hence, from (a), the constant extension map Λ_v of f_v is indeed a lifting of λ_v. ∎

Proposition 3.1(b) asserts that, under hypothesis (Loc), condition (λ/Lift) holds locally. Does it follow that there exists a global lifting Λ: $G(K) \to N/G$? This first obstruction is studied in §4. In the positive case, i.e., when the map λ does have some lifting Λ over K, the next question is: does there exist a K-model with *this* map Λ as constant extension map (in Galois closure)? That question is the topic of §3.2. Finally, there may be several liftings Λ of λ and none is fixed *a priori*. The original question is: does there exist a K-model (with *some* map Λ as constant extension map)? That question is considered in §3.3, which concludes our study of the local-to-global principle.

3.2. THE LOCAL-TO-GLOBAL PRINCIPLE WITH Λ FIXED. We assume in this paragraph that condition (λ/Lift) holds over K and we fix a specific lifting Λ: $G(K) \to N/G$ of $\lambda : G(K) \to N/CG$. For each place $v \in M_K$, let Λ_v: $G(K_v) \to N/G$, obtained by composing Λ with the natural map r_v: $G(K_v) \to G(K)$. The map Λ_v is a lifting of $\lambda_v = \lambda r_v$. In this paragraph, we assume that

(Loc+) for each $v \in M_K$, f has a model f_v with Λ_v as constant extension map.

In other words, a "possible" extension of constants $\widehat{K}/K$ (in Galois closure) is fixed and, for each $v \in M_K$, the cover is assumed to be defined over K_v with $\widehat{K}K_v/K_v$ as extension of constants (in Galois closure).

[DbDo] provides the following characterization of the obstruction to the field of moduli K being a field of definition when the constant extension map Λ is fixed. To the homomorphism Λ: $G(K) \to N/G$ can be associated a 2-cocycle $\Omega_\Lambda \in$

$H^2(K, Z(G), L)$ for a certain action L of $G(K)$ on $Z(G)$ (explicitly described in [DbDo]), with the following property.

PROPOSITION 3.2: *The element Ω_Λ is trivial in $H^2(K, Z(G), L)$ if and only if there exists a K-model of the (G-)cover f with constant extension map (in Galois closure) equal to the map Λ.*

This result corresponds to conclusion (d) of Main Theorem (II) of [DbDo] (with $\theta = 1$). Furthermore, it is clear from the definition of the 2-cocycle Ω_Λ, which is explicit in [DbDo] (see Main Theorem (II) (b)), that it behaves well with extension of scalars. More precisely, let L_v be the action obtained from L by composing with r_v: $G(K_v) \to G(K)$. Then Ω_Λ regarded as an element of $H^2(K_v, Z(G), L_v)$ by extending the scalars coincides with the 2-cocycle Ω_{Λ_v}. Thus we have

PROPOSITION 3.3: *Under hypothesis* (Loc+) *the 2-cocycle Ω_Λ lies in the kernel of the natural map*

(LocGlob) $$H^2(K, Z(G), L) \to \prod_{v \in M_K} H^2(K_v, Z(G), L_v).$$

This reduces the problem to studying the injectivity of this local-global map. This is a difficult problem. There are few positive general results but also few counter-examples (see [Se; Ch.III]). There are, however, two situations where some results are available.

PROPOSITION 3.4: *Consider the case that the abelian group $Z(G)$ is cyclic of order n (to which one may always reduce). Then the local-global map* (LocGlob) *is injective in each of the two following situations:*

(a) *The action L is the cyclotomic action of $G(K)$ on μ_n and K is a number field.*
(b) *The field K is a number field for which the special case of Grunwald's theorem below cannot occur (e.g. $K = \mathbb{Q}$) and the action L is the trivial action. The latter is true if elements of $Z(G)$ commute with those of N, i.e., if $Z(G) \subset Z(N)$.*

Special Case of Grunwald's Theorem: For each integer $r > 0$, ζ_r is a primitive 2^rth root of 1 and $\eta_r = \zeta_r + \zeta_r^{-1}$. Then denote by s the smallest integer such that $\eta_s \in K$ and $\eta_{s+1} \notin K$. The special case is defined by these three simultaneous conditions:

1. -1, $2+\eta_s$, $-(2+\eta_s)$ are non-squares in K.

2. For each place $v \in M_K$ of K dividing 2, at least one out of the elements -1, $2+\eta_s$, $-(2+\eta_s)$ is a square in K_v.

3. The abelian group $Z(G)$ contains an element of order a multiple of 2^t with $t > s$.

If $K = \mathbb{Q}$, then $s = 2$ and $\eta_s = 0$. Since -1, 2 and -2 are non-squares in $\mathbb{Q}_2$, condition 2. is not satisfied. Therefore, the special case cannot occur if $K = \mathbb{Q}$. Similarly, the special case cannot occur if K contains $\sqrt{-1}$ or $\sqrt{-2}$ or if $Z(G)$ is of odd order.

Proof: (a) The group $H^2(K, \mu_n)$ identifies with the subgroup of elements of order n of the Brauer group $\mathrm{Br}(K)$. The result then follows from the injectivity of the local-global map on Brauer groups

$$\mathrm{Br}(K) \to \prod_{v \in M_K} \mathrm{Br}(K_v).$$

(b) follows from Grunwald–Wang's theorem [ArTa; p. 96] conjoined with Tate–Poitou's theorem [Se; II-§6.3] (see [DbDo; Proof of Theorem 3.8] for more details). It remains to show that L is the trivial action if $Z(G) \subset Z(N)$. This follows from this explicit description of the action L [DbDo; §3.2]: action by conjugation of N over $Z(G)$ factors through N/G, compose the induced action of N/G with the map Λ to get the action L. ∎

We will focus on the situation L is the trivial action. The G-cover situation is a special case of it (since $N = G$). Furthermore, in the case of G-covers, Λ is by definition the trivial map; that is in this case the only lifting of λ. As a consequence of Propositions 3.3 and 3.4, we obtain

Corollary 3.5 ([DbDo; Theorem 3.8]): *A G-cover $f\colon X \to B$ over $\overline{\mathbb{Q}}$ is defined over $\mathbb{Q}$ if and only if it is defined over $\mathbb{Q}_p$ for each prime p (including the prime at infinity). More generally, the same conclusion holds over a number field K for which the special case of Grunwald's theorem cannot occur.*

3.3. The local-to-global principle for varying Λ. In this paragraph the (G-)cover f still satisfies the local hypothesis (Loc). In particular K is the field of moduli of f. As in §3.2 we assume that condition (λ/Lift) holds over K but we no longer fix a lifting of λ: $G(K) \to N/CG$. For mere covers there may indeed be several liftings Λ of λ. We first explain why that makes the local-to-global principle unlikely in general.

If the mere cover f has a model f_K over K, the mere covers $\tilde{f}_K \otimes_K K_v$ have the property that the associated constant extension maps $G(K_v) \to N/G$ all come from an identical global lifting Λ: $G(K) \to N/G$ of λ. Now if we only assume that f has a local model f_v over K_v for each place $v \in M_K$, one can hardly expect *a priori* that the associated constant extension maps Λ_v all come from the same global Λ. Possibly can we hope that local existence of liftings Λ_v for each place $v \in M_K$ implies the existence of a global lifting Λ. But this Λ may then not induce each of the Λ_v's. In other words, even if there is a global Λ, the models f_v over K_v of f may not be models over K_v for this specific Λ. Equivalently, the local extensions of constants $\widehat{K_v}/K_v$ in the Galois closure of f_v ($v \in M_K$) may not come by extension of scalars from an identical global extension $\widehat{K}/K$. However, we do not have any counter-example yet (see also [Db; Remark 5.4] for some "possible" counter-examples).

Denote by Δ the set of all possible liftings of λ. It seems that only with a better control on the set of all obstructions Ω_Λ ($\Lambda \in \Delta$) will we be able to prove some local-to-global principle for mere covers. Our previous paper [DbDo] provides the following extra information.

Fix a lifting Λ: $G(K) \to N/G$ of λ and let L_Λ be the action of $G(K)$ on $C/Z(G)$ obtained by composing Λ: $G(K) \to N/G$ with the conjugation of N/G on $CG/G \simeq C/Z(G)$. Denote by δ^1 the coboundary operator

$$H^1(K, C/Z(G), L_\Lambda) \to H^2(K, Z(G), L)$$

given by the exact sequence (the kernel $Z(G)$ of which is abelian and central)

$$1 \to Z(G) \to C \to C/Z(G) \to 1.$$

PROPOSITION 3.6: *Assume that K is the field of moduli of the (G-)cover f and that condition (λ/Lift) holds.*

(a) *For each $\Lambda' \in \Delta$, we have*

$$\Omega_{\Lambda'} \cdot \Omega_\Lambda^{-1} \in \delta^1 \left(H^1(K, C/Z(G), L_\Lambda)\right).$$

(b) *In particular, the field of moduli K is a field of definition of the (G-)cover f if and only if the 2-cocycle Ω_Λ satisfies this condition:*

$$\Omega_\Lambda^{-1} \in \delta^1 \left(H^1(K, C/Z(G), L_\Lambda)\right).$$

Proof: Proposition 3.6(a) readily follows from Main Theorem (II)(c) and Proposition 4.5 of [DbDo]. For the (b) part, observe that the field of moduli K is a field of definition if and only if it is a field of definition for some constant extension map $\Lambda' \in \Delta$. The desired result then follows straightforwardly from Proposition 3.6(a) and Proposition 3.2. Proposition 3.6(b) also corresponds to conclusion (e) of Main Theorem (II) of [DbDo]. ∎

Under condition (λ/Lift), conditions (Loc) and (Glob) can be rewritten as:

(Loc) For each place $v \in M_K$, $\Omega_{\Lambda_v}^{-1} \in \delta^1\left(H^1(K_v, C/Z(G), L_{\Lambda_v})\right) \subset H^2(K_v, Z(G), L_v)$.

(Glob) $\Omega_\Lambda^{-1} \in \delta^1\left(H^1(K, C/Z(G), L_\Lambda)\right) \subset H^2(K, Z(G), L)$.

Assume for a moment that the centralizer C is an abelian group. Then sets $H^2(k, C, -)$ are groups and the coboundary operator is a group homomorphism. Furthermore, we have a commutative diagram

$$(1)\qquad \begin{array}{ccc} \dfrac{H^2(K,Z(G))}{\delta^1(H^1(K,C/Z(G)))} & \longrightarrow & \prod_{v\in M_K} \dfrac{H^2(K_v,Z(G))}{\delta^1(H^1(K_v,C/Z(G)))} \\ \big\downarrow & & \big\downarrow \\ H^2(K,C) & \longrightarrow & \prod_{v\in M_K} H^2(K_v,C) \end{array}$$

where vertical arrows are injective. If the lower map is injective, then so is the upper one. In that case then, we do have (Loc)$\Rightarrow$(Glob). This shows that the local-to-global principle holds if the centralizer C is an abelian group, contained in the center $Z(N)$ of N (in order to make the actions L and L_Λ trivial), and if K is a number field for which the special case of Grunwald's theorem cannot occur.

Let us return to general case, i.e., "C not abelian", which is more difficult for it involves non-abelian cohomology for which there are no such nice properties as the coboundary operator being a homomorphism, etc. However, $H^2(k, C, -)$ are defined as sets (with a free transitive action of $H^2(k, Z(C), -)$ if they are non-empty). It would be interesting to translate the problem in cohomological terms as in the abelian case. There is a special case where this is possible, which contains the preceding case "C abelian". The following result is our more general answer to the local-to-global problem.

THEOREM 3.7: *Let f be a (G-)cover defined over K_v for each place $v \in M_K$. Assume that condition (λ/Lift) holds over K and that the center $Z(G)$ is a direct summand of the centralizer C.*

(a) *Then all 2-cocycles* Ω_Λ*, where* $\Lambda\colon G(K) \to N/G$ *ranges over the set* Δ *of all lifts of* $\lambda\colon G(K) \to N/CG$*, are equal to the same 2-cocycle* Ω*. This 2-cocycle* Ω *lies in the kernel of the local-global map*

$$\text{(LocGlob)} \qquad H^2(K, Z(G), L) \to \prod_{v \in M_K} H^2(K, Z(G), L_v)$$

and has the following property: the (G-)cover f *is defined over* K *if and only if the 2-cocycle* Ω *is trivial in* $H^2(K, Z(G), L)$*.*

(b) *In particular, the local-to-global principle holds if, in addition,* $Z(G) \subset Z(N)$ *and if* K *is a number field for which the special case of Grunwald's theorem cannot occur.*

Proof: For each $\theta \in H^1(K, C/Z(G), L_\Lambda)$, $\delta^1(\theta)$ is the obstruction to the possibility of lifting θ up to a 1-cocycle $\tilde{\theta} \in H^1(K, C, L_\Lambda)$. If $Z(G)$ is a direct summand of the centralizer C, this obstruction always vanishes. Conclude that sets $\delta^1\left(H^1(K, C/Z(G), L_\Lambda)\right)$ consist of the single trivial element 1. It follows from Proposition 3.6(a) that all 2-cocycles Ω_Λ $(\Lambda \in \Delta)$ are the same 2-cocycle Ω. From Proposition 3.6(b), the field of moduli K is a field of definition of the (G-)cover f if and only if Ω is trivial in $H^2(K, Z(G), L)$. Finally from Proposition 3.3, Ω lies in the kernel of the map (LocGlob). The (b) part of Theorem 3.7 then readily follows from Proposition 3.4(b). ∎

Remark 3.8: We explained that what causes the problem in general is the possibility that there are several liftings Λ of λ. What the assumption "$Z(G)$ is a direct summand of C" did was to insure that all the corresponding 2-cocycles Ω_Λ $(\Lambda \in \Delta)$ are equal in $H^2(K, Z(G), L)$.

3.4. A VARIANT OF THE LOCAL-TO-GLOBAL PROBLEM (after an idea of Lenstra). Consider this slight change of the local-to-global problem: replace in the hypothesis the phrase "for all places $v \in M_K$" by "for all places $v \in M_K$ but one". Does Theorem 3.7(b) still hold with that change? The answer is "Yes". There is only one argument to add to the proof. One should show that an element $\Omega \in H^2(K, Z(G))$ (for the trivial action) that vanishes in $H^2(K_v, Z(G))$ for all places $v \in M_K$ but one, necessarily also vanishes for the missing place, and so lies indeed in the kernel of the map (LocGlob). One may reduce to the case $Z(G) = \mathbb{Z}/n\mathbb{Z}$.

The result is then classical in the case that K contains nth roots of 1. Indeed, $H^2(K_v, \mathbb{Z}/n\mathbb{Z})$ can be viewed as the subgroup $\mathrm{Br}_n(K)$ of elements of order n in the Brauer group $\mathrm{Br}(K)$ of K. Now each Brauer group $\mathrm{Br}(K_v)$ can be identified to $\mathbb{Q}/\mathbb{Z}$ with this property: the sum of all local terms induced in each $\mathrm{Br}(K_v) = \mathbb{Q}/\mathbb{Z}$ by a given element of $\mathrm{Br}(K)$ is 0. Consequently, all but exactly one of these local terms cannot be 0.

The proof is a little more complicated when K does not contain the nth roots of 1. From [Se; Th. 2 p. 108], for each place $v \in M_K$, each group $H^2(K_v, \mathbb{Z}/n\mathbb{Z})$ can be identified to the dual $H^0(K_v, \mu_n)'$ of $H^0(K_v, \mu_n)$. Given a field F, denote the group of all nth roots in F by $\mu_n(F)$. Also set $\mu_n(\overline{K}) = \mu_n$. Then we have, for each place $v \in M_K$, $H^0(K_v, \mu_n) = \mu_n(K_v) = \mu_n^{e_v}$ for some divisor e_v of n and

$$H^0(K_v, \mu_n)' = (\mu_n(K_v))' = (\mu_n^{e_v})' = e_v\mathbb{Z}/n\mathbb{Z} \subset \mathbb{Z}/nZ.$$

This identification of $H^2(K_v, \mathbb{Z}/n\mathbb{Z})$ with the subgroup $e_v\mathbb{Z}/n\mathbb{Z}$ of $\mathbb{Z}/nZ$ has the same property as above, namely, the sum of all the local terms induced in each $H^2(K_v, \mathbb{Z}/n\mathbb{Z}) \subset \mathbb{Z}/n\mathbb{Z}$ by a given element of $H^2(K, \mathbb{Z}/n\mathbb{Z})$ is 0. Consequently, all but exactly one of these local terms cannot be 0.

4. Condition (λ/Lift)

Keep the notation of §2 and §3. From Proposition 3.1, under the hypothesis (Loc), i.e., "the (G-)cover f is defined over K_v for each $v \in M_K$", condition (λ/Lift) necessarily holds locally, i.e., over each K_v ($v \in M_K$). The question of concern in this section is whether it can be inferred that condition (λ/Lift) holds globally, i.e., that there is a global lifting $\Lambda: G(K) \to N/G$ of $\lambda: G(K) \to N/CG$? This is a local-to-global problem similar to the one studied in §3. [DbDo] gives the following cohomological criterion for condition (λ/Lift). There is a minimal necessary assumption in this criterion, called (Band/Rep), and which we will explain right after Proposition 4.1.

PROPOSITION 4.1: *Assume that K is the field of moduli of the (G-)cover f and that condition* (Band/Rep) *holds. Then there exists an action χ of $G(K)$ on $Z(C/Z(G))$ and a 2-cocycle $\omega \in H^2(K, Z(C/Z(G)), \chi)$ such that condition* (λ/Lift) *is equivalent to the condition*

$$\omega^{-1} \in \delta^1 \left(H^1(K, \mathrm{Inn}(C/Z(G)), \tilde{\ell})\right) \subset H^2(K, Z(C/Z(G)), \chi)$$

where $\mathrm{Inn}(C/Z(G)))$ *denotes the inner automorphism group of* $C/Z(G)$, $\tilde{\ell}$ *is a certain action of* $G(K)$ *on* $\mathrm{Inn}(C/Z(G))$ *(described below) and* δ^1 *is the* *coboundary operator*

$$H^1(K, \mathrm{Inn}(C/Z(G)), \tilde{\ell}) \to H^2(K, Z(C/Z(G)), \chi).$$

This result corresponds to Theorem 4.7 of [DbDo] where the action χ and the 2-cocycle ω are explicitly described.

For example, condition (λ/Lift) holds if condition (Band/Rep) holds and $C/Z(G)$ is a centerless group. We explain now condition (Band/Rep). Consider the exact sequence

$$1 \to CG/G \to N/G \to N/CG \to 1.$$

There is no natural action of N/CG on the kernel $CG/G \simeq C/Z(G)$, but only an **outer action**, i.e., a homomorphism $\overline{\kappa}$: $N/CG \to \mathrm{Out}(CG/G)$ from N/CG to the outer automorphism group

$$\mathrm{Out}(CG/G) = \mathrm{Aut}(CG/G)/\,\mathrm{Inn}(CG/G).$$

Following Giraud's terminology [Gi], the map $\overline{\kappa}\lambda$: $G(K) \to \mathrm{Out}(CG/G)$ is called the **band** of the problem. A necessary condition for condition (λ/Lift) is that the band is **representable**, i.e., that the outer action $\overline{\kappa}\lambda$: $G(K) \to \mathrm{Out}(CG/G)$ can be lifted to a real action ℓ: $G(K) \to \mathrm{Aut}(CG/G)$. That condition is condition (Band/Rep). It holds, for example, if $\mathrm{Inn}(CG/G)$ has a complement in $\mathrm{Aut}(CG/G)$.

The action $\tilde{\ell}$ of $G(K)$ on $\mathrm{Inn}(C/Z(G))$ involved in the statement of Proposition 4.1 is obtained by composing ℓ: $G(K) \to \mathrm{Aut}(CG/G)$ with the conjugation of $\mathrm{Aut}(CG/G)$ on $\mathrm{Inn}(CG/G)$). The condition

$$\text{“}\omega^{-1} \in \delta^1 \left(H^1(K, \mathrm{Inn}(C/Z(G)), \tilde{\ell})\right)\text{”}$$

of Proposition 4.1 does not depend on the choice of the action ℓ lifting the outer action $\overline{\kappa}\lambda$.

Under condition (Band/Rep), the cohomological formulation of the problem of this section is very similar to that of §3. The same arguments apply to prove this result.

THEOREM 4.2: *Let f be a (G-)cover satisfying condition* (λ/Lift) *over K_v for each $v \in M_K$ (e.g. f is defined over K_v for each place $v \in M_K$). Assume that condition* (Band/Rep) *holds over K (e.g.* Inn(CG/G) *has a complement in* Aut(CG/G)*) and that $Z(C/Z(G))$ is a direct summand of the group $C/Z(G)$.*

(a) *Then there exists a 2-cocycle ω lying in the kernel of the local-global map*

$$H^2(K, Z(C/Z(G)), \chi) \to \prod_{v \in M_K} H^2(K, Z(C/Z(G)), \chi_v)$$

with the following property: Condition (λ/Lift) *holds over K if and only if the 2-cocycle ω is trivial.*

(b) *In particular, condition* (λ/Lift) *holds if, in addition, $Z(CG/G) \subset Z(N/G)$ and if K is a number field for which the special case of Grunwald's theorem cannot occur.*

Proof: Similarly to the proof of Theorem 3.7, the assumption "$Z(C/Z(G))$ is a direct summand of the group $C/Z(G)$" guarantees that the set $\delta^1\left(H^1(K, \mathrm{Inn}(C/Z(G)), \tilde{\ell})\right)$ involved in Proposition 4.1 consists of the single trivial element 1. Consequently, the vanishing of the 2-cocycle ω of Proposition 4.1 is an *if and only if* condition for (λ/Lift) to hold. Next, it follows from the definition of ω (given in [DbDo]) that it behaves well with extensions of scalars. In particular, regarded as an element of $H^2(K_v, Z(C/Z(G)), \chi_v)$ by extending the scalars, the 2-cocycle ω coincides with the 2-cocycle ω_v associated to the (G-)cover f_v. Conclude from the assumption "(λ/Lift) holds over K_v for each $v \in M_K$" that ω lies in the kernel of the natural local-global map of Theorem 4.2(a). This proves Theorem 4.2(a). The (b) part of Theorem 4.2 then readily follows from Proposition 3.4(b). ∎

5. A global-to-local principle

In this section, we assume that K is a global field (i.e., a number field or a function field of a curve over a finite field), or, more generally, a field with the product formula satisfying condition (**) below (condition (*) is not necessary here):

(**) For all but finitely many finite places $v \in M_K$, the Galois group $G(K_v^{ur}/K_v)$ of the unramified maximal extension of K_v is a projective profinite group; in particular, groups $H^2(G(K_v^{ur}/K_v), -, -)$ are trivial.

Condition (**) is also satisfied by function fields $\overline{k}(C)$ of a curve or a surface over an algebraically closed field. This section is devoted to the following global-to-local principle.

THEOREM 5.1: *Let $f\colon X \to B$ be a (G-)cover a priori defined over K_s. Suppose that K is the field of moduli of the (G-)cover f. Then the cover can be defined over all but finitely many completions K_v of K.*

This result was proved in [Db; Theorem 8.1] in the case of G-covers and mere covers of $\mathbb{P}^1$ over $\overline{\mathbb{Q}}$. The proof was effective in the sense that an explicit bound for the exceptional places can be derived from it [Sa]. Extending this proof to the situation of covers of more general spaces than $\mathbb{P}^1$ presents some difficulties. But at some place of the proof, a more general argument can be used that avoids these difficulties. This more general argument was explained to us by H. Lenstra. It is this proof with Lenstra's argument that we give here. The proof, however, is no longer effective. The main idea is the same for all these proofs: for all but finitely many places, find a model over the unramified closure K_v^{ur} of K_v and then take advantage of the projectivity of the Galois group $G(K_v^{ur}/K_v)$ to descend to K_v. A version of this strategy was first considered in Dew's thesis [Dew].

Proof: There exists a finite Galois extension F/K satisfying these two conditions:

(i) the (G-)cover f has a F-model f_F over F,

(ii) for each $\tau \in G(F/K)$, the covers f_F and f_F^{τ} are isomorphic over F.

(For example, pick a finite Galois extension F_o of K over which the cover f is defined and take for F the Galois closure over K of the field generated by F_o and the coefficients of isomorphisms between the cover f and its distinct conjugates under $G(F_o/K)$.) Condition (ii) insures that the field of moduli of the (G-)cover f_F relative to the extension F/K is equal to K (see [DbDo; §2.7] for a precise definition of *relative field of moduli*).

The extension F/K is unramified at all but finitely many places $v \in M_K$. For such places v, set $f_v = f_F \otimes_F K_v^{ur}$: this is a (G-)cover over K_v^{ur} with field of moduli relative to the extension K_v^{ur}/K_v equal to K_v. From Corollary 3.3 of [DbDo], the field of moduli relative to an extension F/K is a field of definition if the Galois group $G(F/K)$ is projective. Thus conclude from hypothesis (**) that f_v is defined over K_v for all but finitely many places $v \in M_K$. This completes the proof, since clearly the obtained K_v-covers are also K_v-models of the original (G-)cover f. ∎

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